

Rethinking Algorithms for Future Architectures: Communication-Avoiding Algorithms

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- Other members of the ParLab, BEBOP, CACHE, EASI, MAGMA, PLASMA, TOPS projects

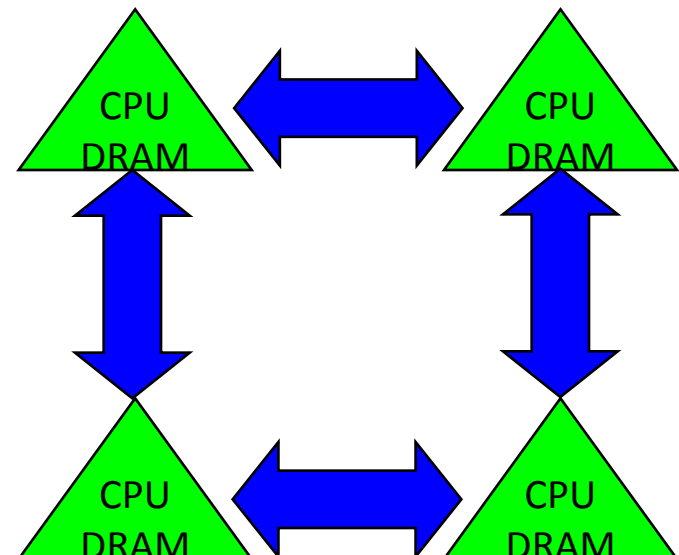
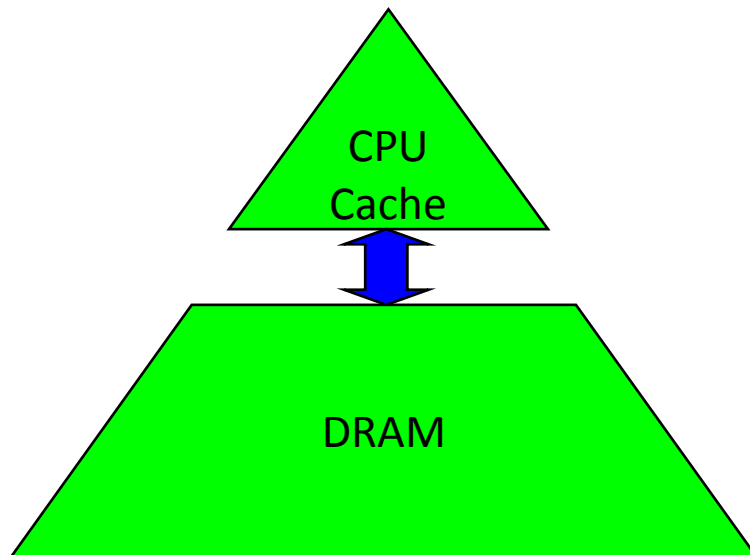
- Supporters

- NSF, DOE, UC Discovery
- Intel, Microsoft, Mathworks, National Instruments, NEC, Nokia, NVIDIA, Samsung, Sun

Why avoid communication? (1/2)

Algorithms have two costs (measured in time or energy):

1. Arithmetic (FLOPS)
2. Communication: moving data between
 - levels of a memory hierarchy (sequential case)
 - processors over a network (parallel case).



Why avoid communication? (2/2)

- Running time of an algorithm is sum of 3 terms:
 - # flops * time_per_flop
 - # words moved / bandwidth
 - # messages * latency } communication
- Time_per_flop \ll 1/ bandwidth \ll latency
 - Gaps growing exponentially with time [FOOSC]

Annual improvements			
Time_per_flop		Bandwidth	Latency
59%	Network	26%	15%
	DRAM	23%	5%

- Goal : reorganize algorithms to *avoid* communication
 - Between all memory hierarchy levels
 - L1 \longleftrightarrow L2 \longleftrightarrow DRAM \longleftrightarrow network, etc
 - Very large speedups possible
 - Energy savings too!

President Obama cites Communication-Avoiding Algorithms in the FY 2012 Department of Energy Budget Request to Congress:

“New Algorithm Improves Performance and Accuracy on Extreme-Scale Computing Systems. **On modern computer architectures, communication between processors takes longer than the performance of a floating point arithmetic operation by a given processor.** ASCR researchers have developed a new method, derived from commonly used linear algebra methods, to **minimize communications between processors and the memory hierarchy, by reformulating the communication patterns specified within the algorithm.** This method has been implemented in the TRILINOS framework, a highly-regarded suite of software, which provides functionality for researchers around the world to solve large scale, complex multi-physics problems.”

FY 2010 Congressional Budget, Volume 4, FY2010 Accomplishments, Advanced Scientific Computing Research (ASCR), pages 65-67.

CA-GMRES (Hoemmen, Mohiyuddin, Yelick, JD)
“Tall-Skinny” QR (Grigori, Hoemmen, Langou, JD)

Outline

- “Direct” Linear Algebra
 - Lower bounds on communication for linear algebra problems like $Ax=b$, least squares, $Ax = \lambda x$, SVD, etc
 - New algorithms that attain these lower bounds
 - *Not* in libraries like Sca/LAPACK (yet!)
 - Large speed-ups possible
 - Implications for architectural scaling
 - How flop rate, bandwidths, latencies, memory sizes need to scale to maintain balance
- Ditto for “Iterative” Linear Algebra

Lower bound for all “direct” linear algebra

- Let M = “fast” memory size (per processor)

$$\#words_moved \text{ (per processor)} = \Omega(\#flops \text{ (per processor)} / M^{1/2})$$

- Parallel case: assume either load or memory balanced
- Holds for
 - Matmul

Lower bound for all “direct” linear algebra

- Let M = “fast” memory size (per processor)

$$\#words_moved \text{ (per processor)} = \Omega(\#flops \text{ (per processor)} / M^{1/2})$$

$$\#messages_sent \geq \#words_moved / largest_message_size$$

- Parallel case: assume either load or memory balanced
- Holds for
 - Matmul, BLAS, LU, QR, eig, SVD, tensor contractions, ...
 - Some whole programs (sequences of these operations, no matter how individual ops are interleaved, eg A^k)
 - Dense and sparse matrices (where $\#flops \ll n^3$)
 - Sequential and parallel algorithms
 - Some graph-theoretic algorithms (eg Floyd-Warshall)

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Can we attain these lower bounds?

- Do conventional dense algorithms as implemented in LAPACK and ScaLAPACK attain these bounds?
 - Mostly not
- If not, are there other algorithms that do?
 - Yes, for much of dense linear algebra
 - New algorithms, with new numerical properties, new ways to encode answers, new data structures
 - Not just loop transformations
- Only a few sparse algorithms so far
- Lots of work in progress...

TSQR: QR of a Tall, Skinny matrix

$$W = \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix}$$

$$\begin{pmatrix} R_{00} \\ R_{10} \\ R_{20} \\ R_{30} \end{pmatrix} = \begin{pmatrix} Q_{01} & R_{01} \\ Q_{11} & R_{11} \end{pmatrix}$$

$$\begin{pmatrix} R_{01} \\ R_{11} \end{pmatrix} = \begin{pmatrix} Q_{02} & R_{02} \end{pmatrix}$$

TSQR: QR of a Tall, Skinny matrix

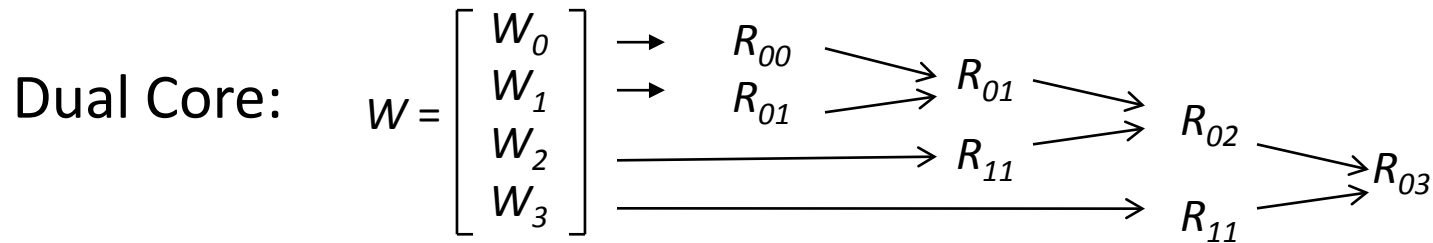
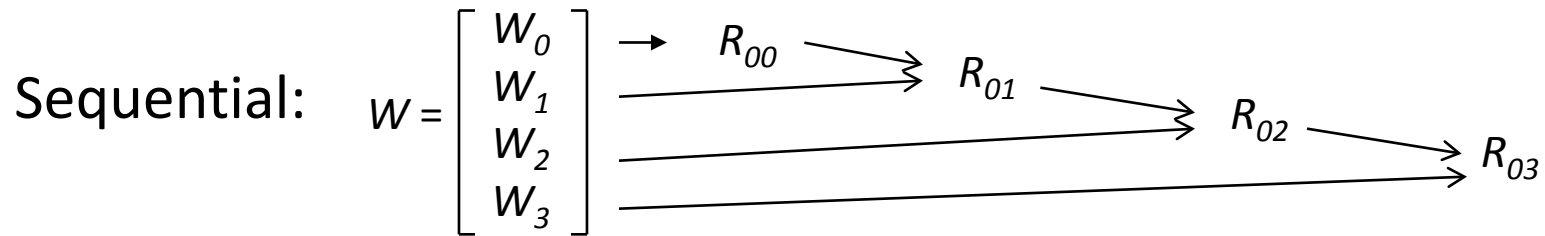
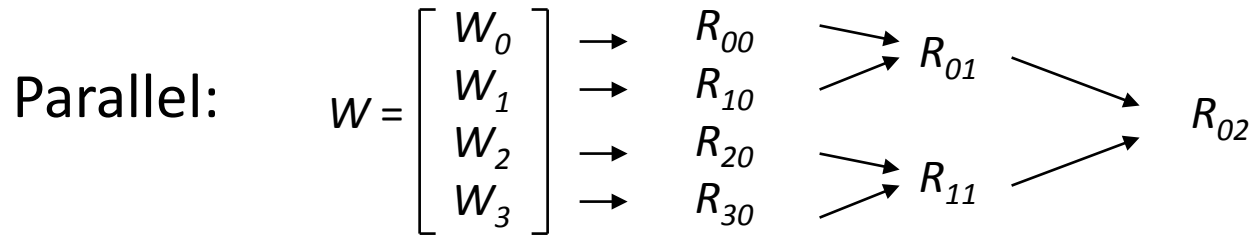
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$$\begin{pmatrix} R_{01} \\ R_{11} \end{pmatrix} = \begin{pmatrix} Q_{02} & R_{02} \end{pmatrix}$$

Output = $\{ Q_{00}, Q_{10}, Q_{20}, Q_{30}, Q_{01}, Q_{11}, Q_{02}, R_{02} \}$

TSQR: An Architecture-Dependent Algorithm



Multicore / Multisocket / Multirack / Multisite / Out-of-core: ?

Can choose reduction tree dynamically

TSQR Performance Results

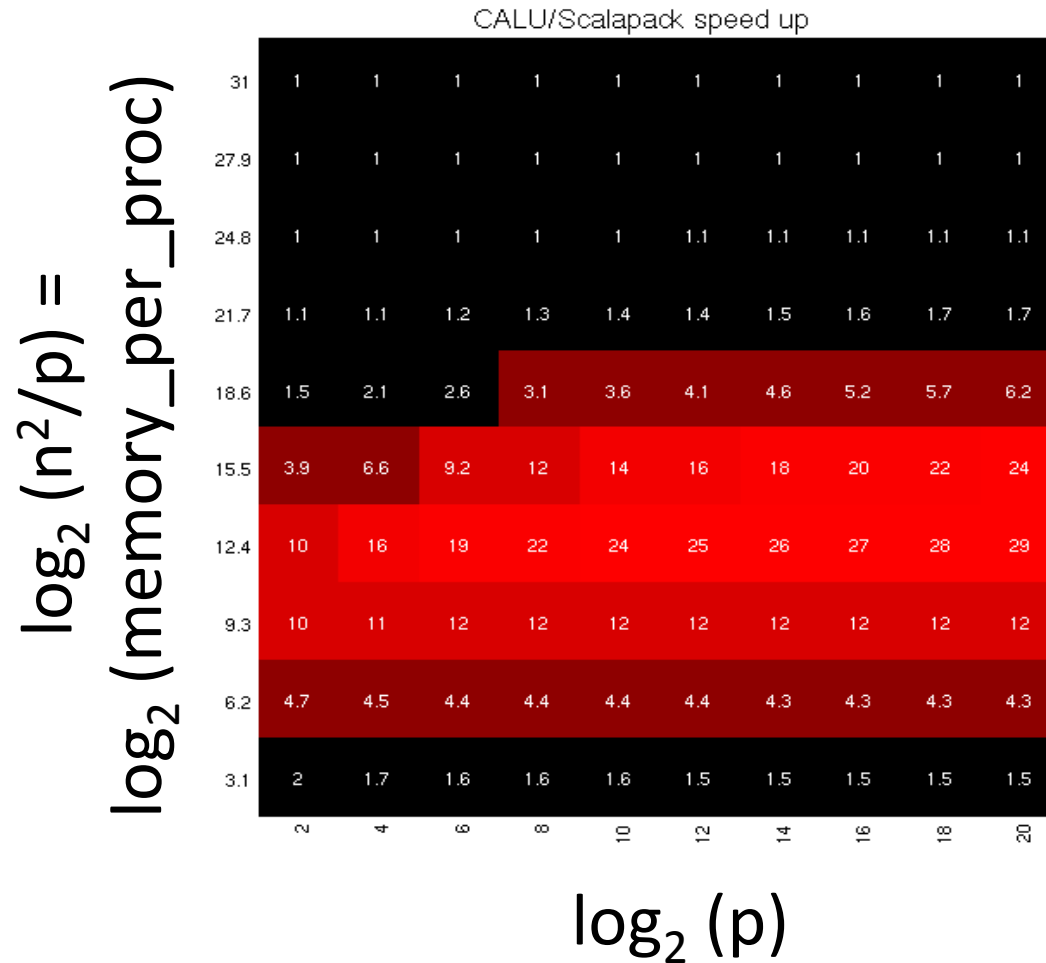
- Parallel
 - Intel Clovertown
 - Up to **8x** speedup (8 core, dual socket, 10M x 10)
 - Pentium III cluster, Dolphin Interconnect, MPICH
 - Up to **6.7x** speedup (16 procs, 100K x 200)
 - BlueGene/L
 - Up to **4x** speedup (32 procs, 1M x 50)
 - Tesla C 2050 / Fermi
 - Up to **13x** (110,592 x 100)
 - Grid – **4x** on 4 cities (Dongarra et al)
 - Cloud – early result – up and running
- Sequential
 - “Infinite speedup” for out-of-Core on PowerPC laptop
 - As little as 2x slowdown vs (predicted) infinite DRAM
 - LAPACK with virtual memory never finished

Exascale Machine Parameters

Source: DOE Exascale Workshop

- $2^{20} \approx 1,000,000$ nodes
- 1024 cores/node (a billion cores!)
- 100 GB/sec interconnect bandwidth
- 400 GB/sec DRAM bandwidth
- 1 microsec interconnect latency
- 50 nanosec memory latency
- 32 Petabytes of memory
- 1/2 GB total L1 on a node

Exascale predicted speedups for Gaussian Elimination: CA-LU vs ScaLAPACK-LU

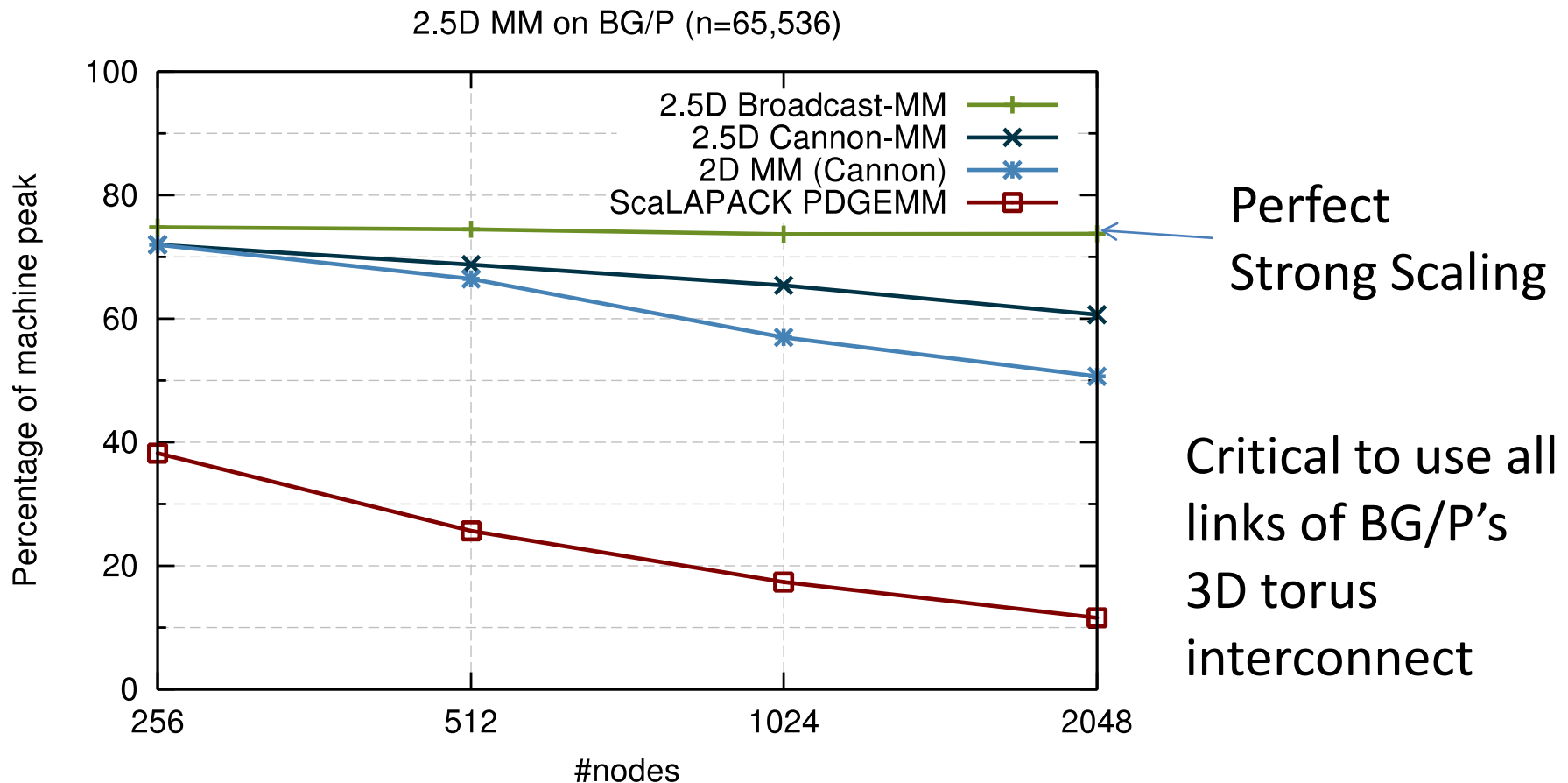


Are we using all the hardware resources?

- Assume $n \times n$ dense matrices on P processors
- Usual approach
 - 1 copy of data \Rightarrow Memory per processor = $M \approx n^2 / P$
 - Recall lower bounds:
 - #words_moved = $\Omega((n^3 / P) / M^{1/2}) = \Omega(n^2 / P^{1/2})$
 - #messages = $\Omega((n^3 / P) / M^{3/2}) = \Omega(P^{1/2})$
 - Attained by **2D algorithms** (many examples)
 - P processors connected in $P^{1/2} \times P^{1/2}$ mesh
 - Each processor owns, computes on a square submatrix
- New approach
 - **Use all available memory**
 - $c > 1$ copies of data \Rightarrow Memory per processor = $M \approx c n^2 / P$
 - Lower bounds get smaller
 - New **2.5D algorithms** can attain new lower bounds
 - P processors in $(P/c)^{1/2} \times (P/c)^{1/2} \times c$ mesh

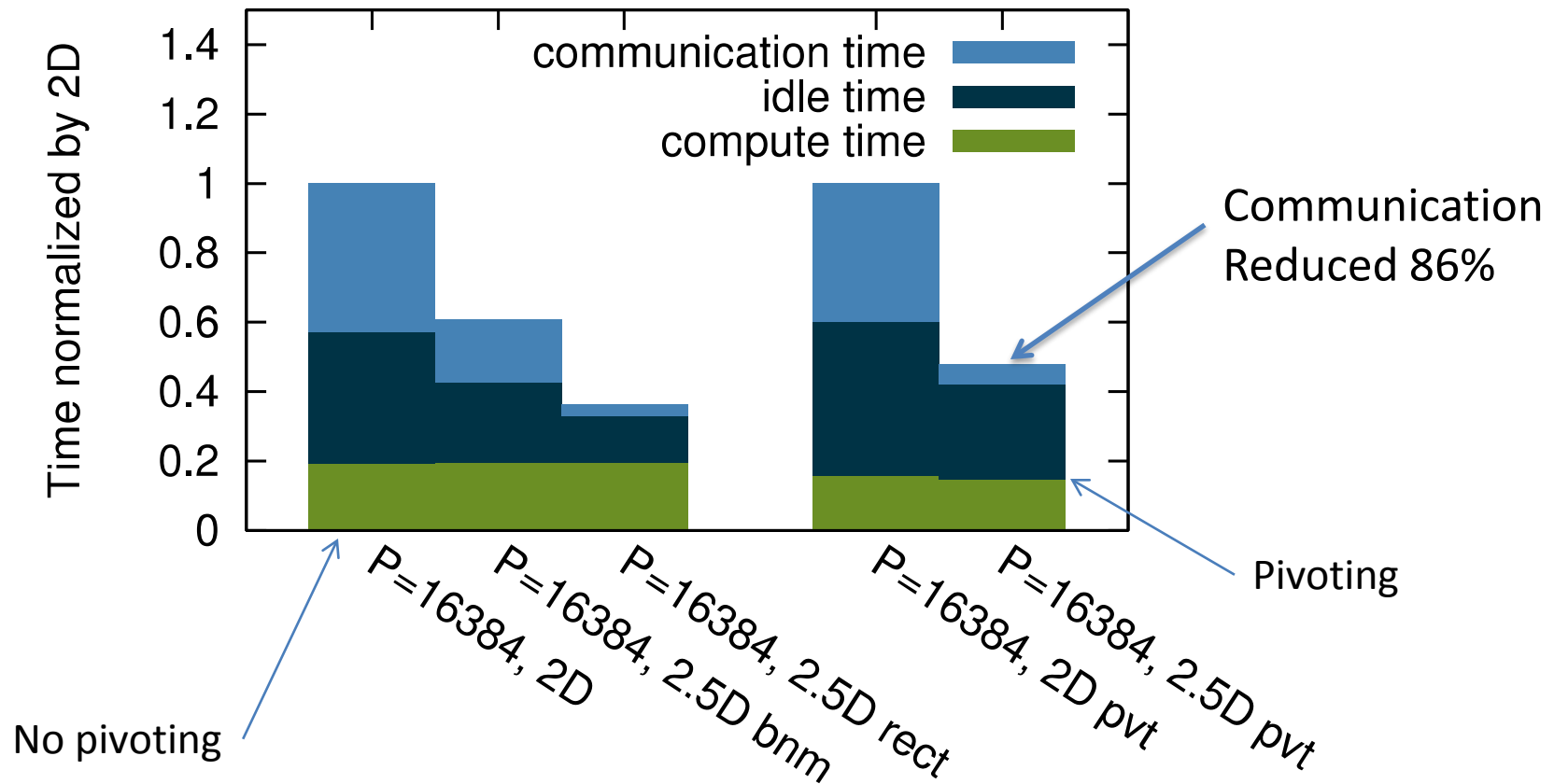
2.5D Matmul versus ScaLAPACK

- 2D algorithms use $P^{1/2} \times P^{1/2}$ mesh and minimal memory
- 2.5D algorithms use $(P/c)^{1/2} \times (P/c)^{1/2} \times c^{1/2}$ mesh and c -fold memory
 - Matmul sends $c^{1/2}$ times fewer words – lower bound
 - Matmul sends $c^{3/2}$ times fewer messages – lower bound



Timing Breakdown for 2D vs 2.5D Gaussian Elimination: How much communication can we avoid?

2.5D LU vs 2D LU on BG/P n=131,072



Distinguished Paper Award, EuroPar'11

Implications for Architectural Scaling

- Machine parameters:
 - γ = seconds per flop (multiply or add)
 - β = reciprocal bandwidth (in seconds)
 - α = latency (in seconds)
 - M = local (fast) memory size
 - P = number of processors
- Goal: relationships among these parameters that guarantees that communication is not the bottleneck for direct linear algebra

Implications for Architectural Scaling

Sequential Case:

- Requirements so that “most” time is spent doing arithmetic on $n \times n$ dense matrices, $n^2 > M$
 - $\gamma M^{1/2} > \approx \beta$ $\gamma M^{1/3} > \approx \beta$ for old algorithms
 - In other words, time to add two rows of largest locally storable square matrix exceeds reciprocal bandwidth
 - $\gamma M^{3/2} > \approx \alpha$ $\gamma M > \approx \alpha$ for old algorithms
 - In other words, time to multiply 2 largest locally storable square matrices exceeds latency
 - Applies to *every* level of memory hierarchy
- Stricter requirements on architecture for old algorithms

Implications for Architectural Scaling

Parallel Case:

- Requirements so that “most” time is spent doing arithmetic on $n \times n$ dense matrices

$$- \gamma (n/p^{1/2}) > \approx \beta \qquad \gamma M^{1/2} > \approx \beta$$

- In other words, time to add two rows of locally stored square matrix exceeds reciprocal bandwidth

$$- \gamma (n/p^{1/2})^3 > \approx \alpha \quad \gamma (n/p^{1/2})^2 > \approx \alpha \quad \gamma M^{3/2} > \approx \alpha$$

- In other words, time to multiply 2 locally stored square matrices exceeds latency

Stricter requirements on architecture for old algorithms

Looser requirements on architecture for 2.5D algorithms

Summary of Direct Linear Algebra

- New lower bounds, optimal algorithms, big speedups in theory and practice
- Lots of other progress, open problems
 - Heterogeneous architectures
 - Extends to case where each processor and link has a different speed (SPAA'11)
 - More dense and sparse algorithms done, underway
 - Extensions to Strassen-like algorithms
 - **Best Paper Award, SPAA'11**
 - Need Autotuning

Avoiding Communication in Iterative Linear Algebra

- k-steps of iterative solver for sparse $Ax=b$ or $Ax=\lambda x$
 - Does k SpMV's with A and starting vector
 - Many such “Krylov Subspace Methods”
- Goal: minimize communication
 - Assume matrix “well-partitioned”
 - Serial implementation
 - Conventional: $O(k)$ moves of data from slow to fast memory
 - **New: $O(1)$ moves of data – optimal**
 - Parallel implementation on p processors
 - Conventional: $O(k \log p)$ messages (k SpMV calls, dot prods)
 - **New: $O(\log p)$ messages - optimal**
- Lots of speed up possible (modeled and measured)
 - Price: some redundant computation

Minimizing Communication of GMRES to solve $Ax=b$

- GMRES: find x in $\text{span}\{b, Ab, \dots, A^k b\}$ minimizing $\|Ax - b\|_2$

Standard GMRES

```
for i=1 to k
  w = A · v(i-1) ... SpMV
  MGS(w, v(0), ..., v(i-1))
  update v(i), H
endfor
solve LSQ problem with H
```

Communication-avoiding GMRES

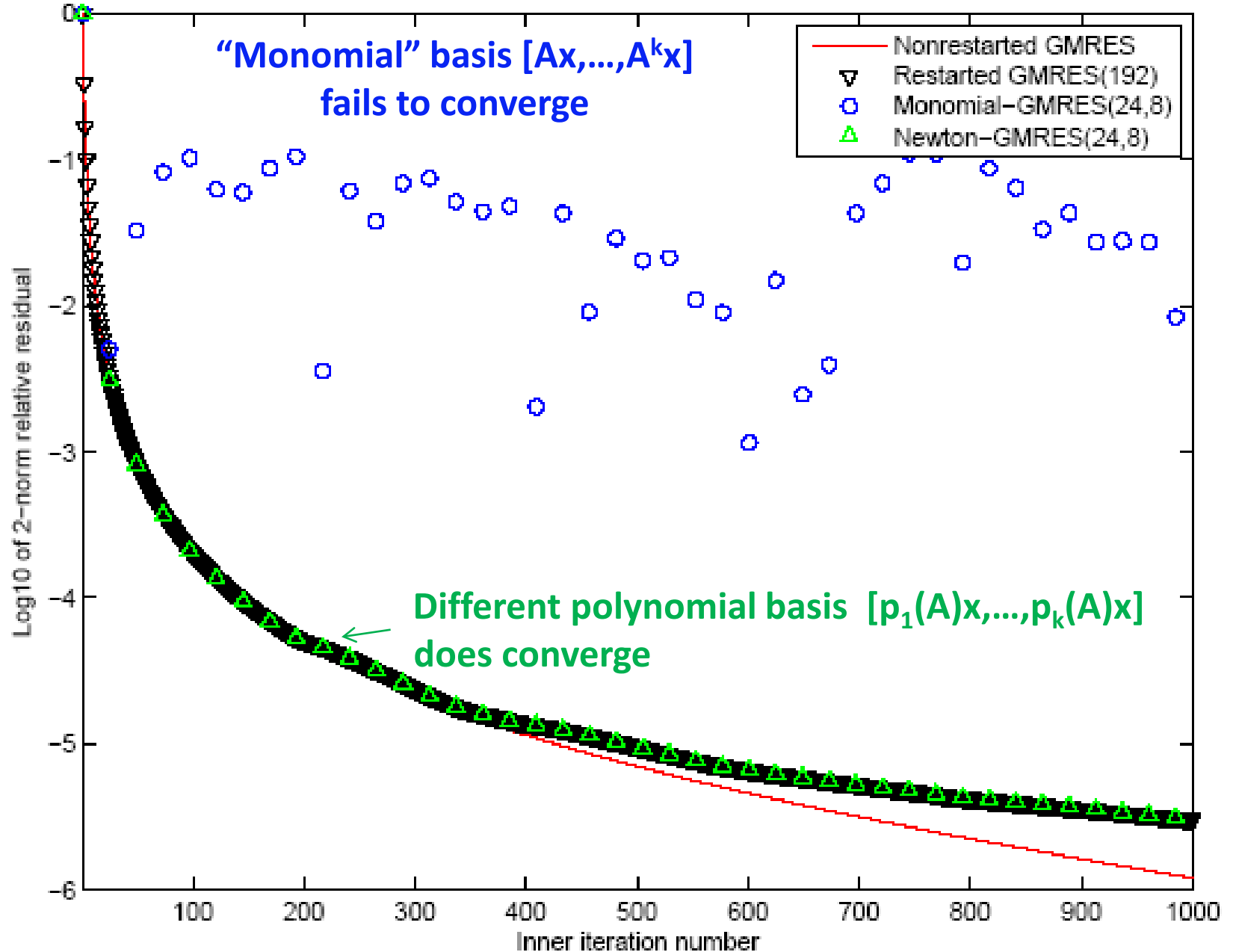
```
W = [ v, Av, A2v, ... , Akv ]
[Q,R] = TSQR(W)
... "Tall Skinny QR"
build H from R
solve LSQ problem with H
```

Sequential case: #words moved decreases by a factor of k

Parallel case: #messages decreases by a factor of k

- **Oops – W from power method, precision lost!**

Matrix diag-cond=1.000000e-11: rel. 2-nrm resid.

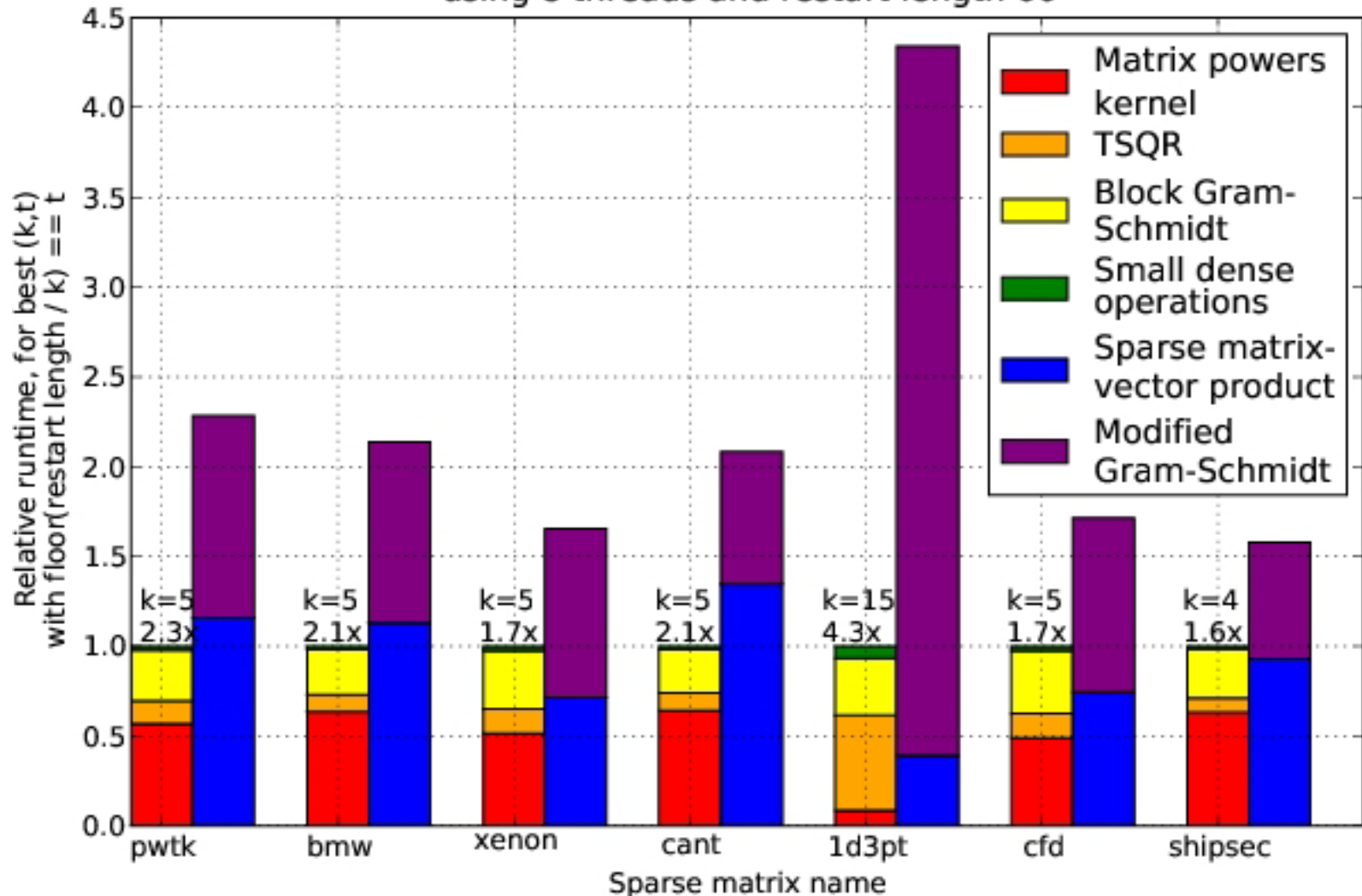


Speed ups of GMRES on 8-core Intel Clovertown

Requires Co-tuning Kernels

[MHDY09]

Runtime per kernel, relative to CA-GMRES(k,t), for all test matrices, using 8 threads and restart length 60



Summary of Iterative Linear Algebra

- New Lower bounds, optimal algorithms, big speedups in theory and practice
- Lots of other progress, open problems
 - Many different algorithms reorganized
 - More underway
 - Architectural scaling rules (as for direct case)
 - Sparse matrices \Rightarrow stricter conditions for scaling
 - Need to recognize stable variants more easily
 - Need Autotuning

For further information

- www.cs.berkeley.edu/~demmel
- Papers
 - bebop.cs.berkeley.edu
 - www.netlib.org/lapack/lawns
- 1-week-short course – slides and video
 - www.ba.cnr.it/ISSNLA2010
- Google “parallel computing course”

Summary

Time to redesign all linear algebra
algorithms and software

And eventually the rest of applied mathematics

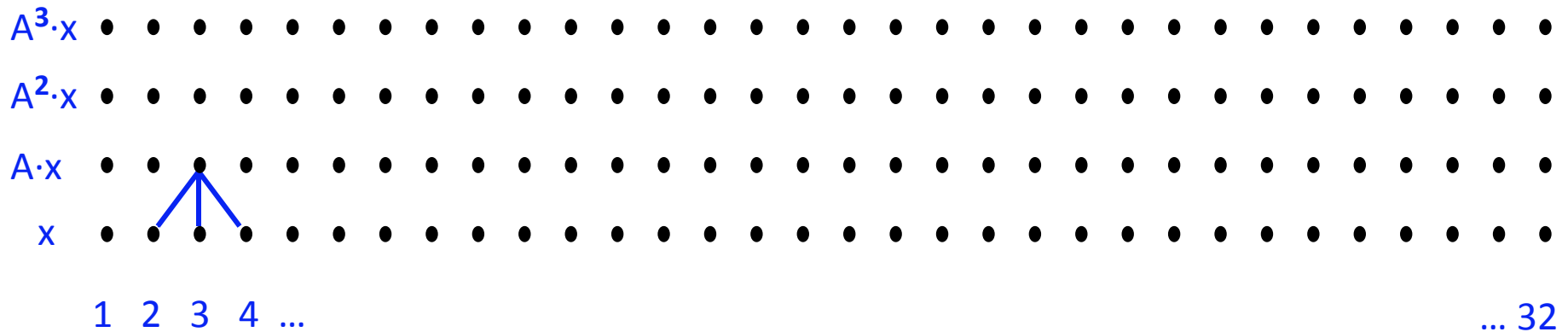
Don't Communic...

EXTRA SLIDES

Communication Avoiding Kernels:

The Matrix Powers Kernel : $[Ax, A^2x, \dots, A^kx]$

- Replace k iterations of $y = A \cdot x$ with $[Ax, A^2x, \dots, A^kx]$

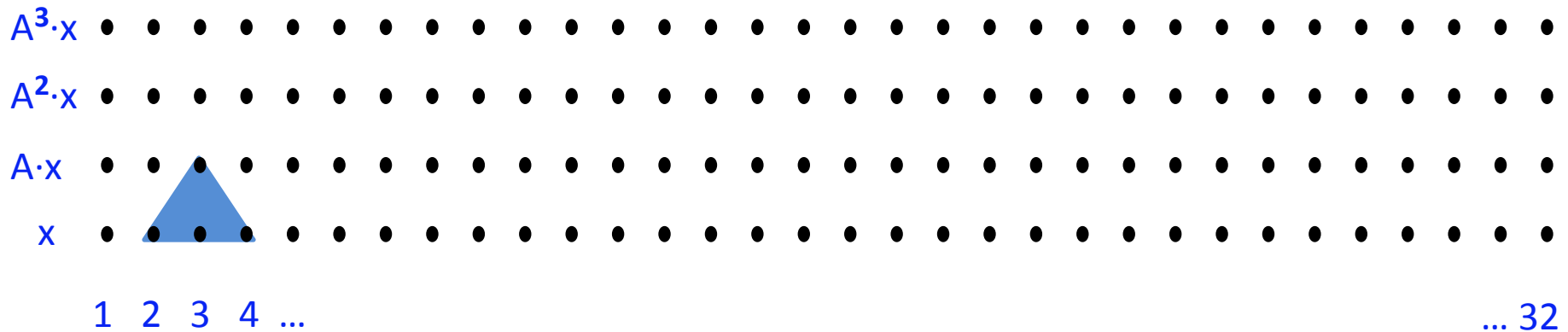


- Example: A tridiagonal, $n=32$, $k=3$

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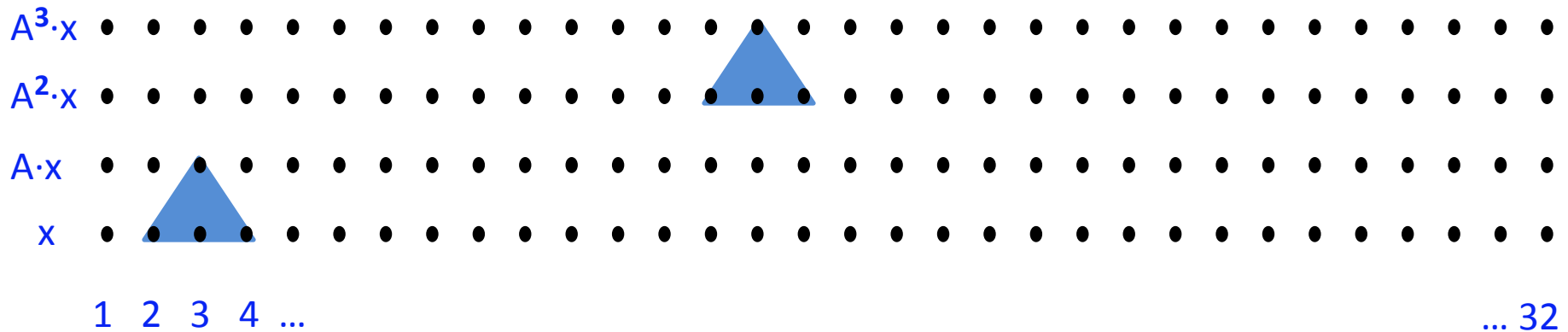


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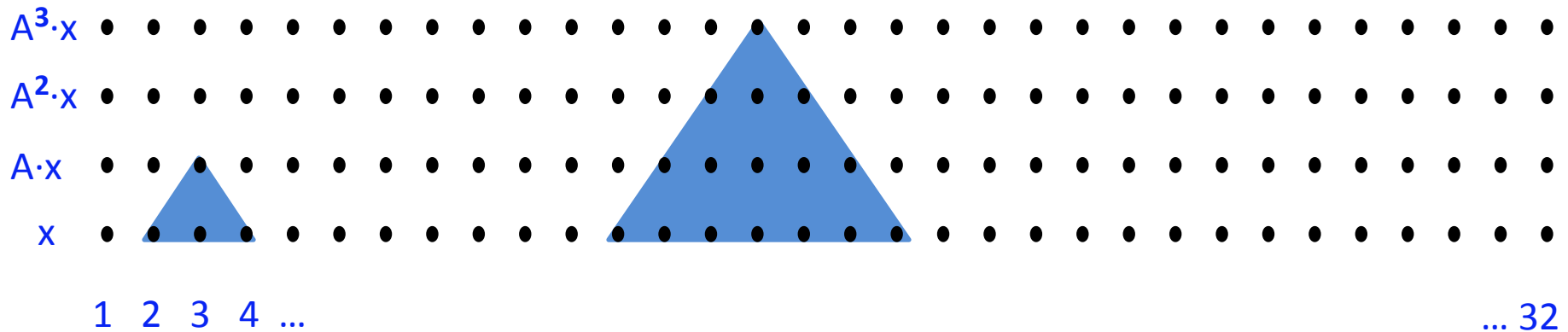


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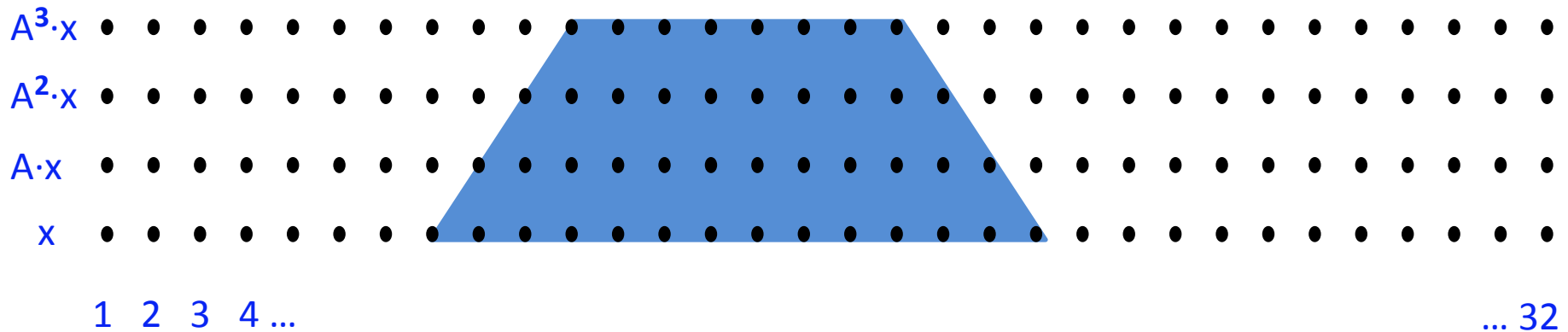


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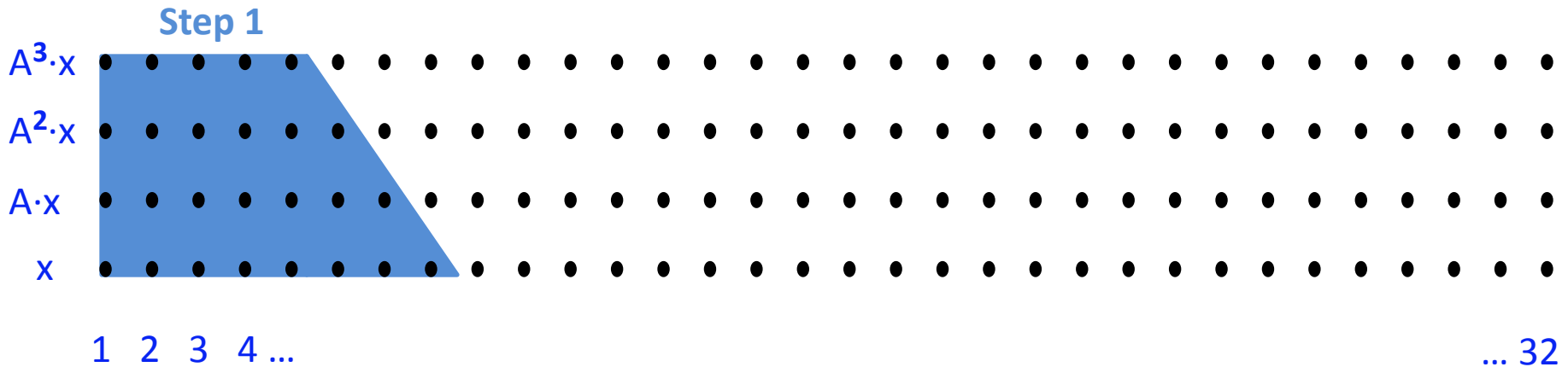


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- Sequential Algorithm

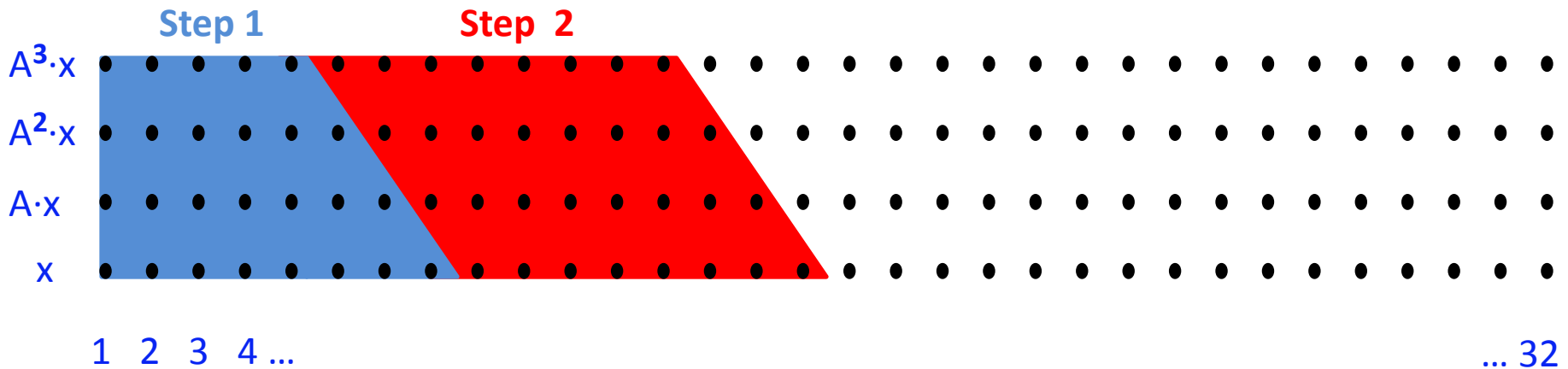


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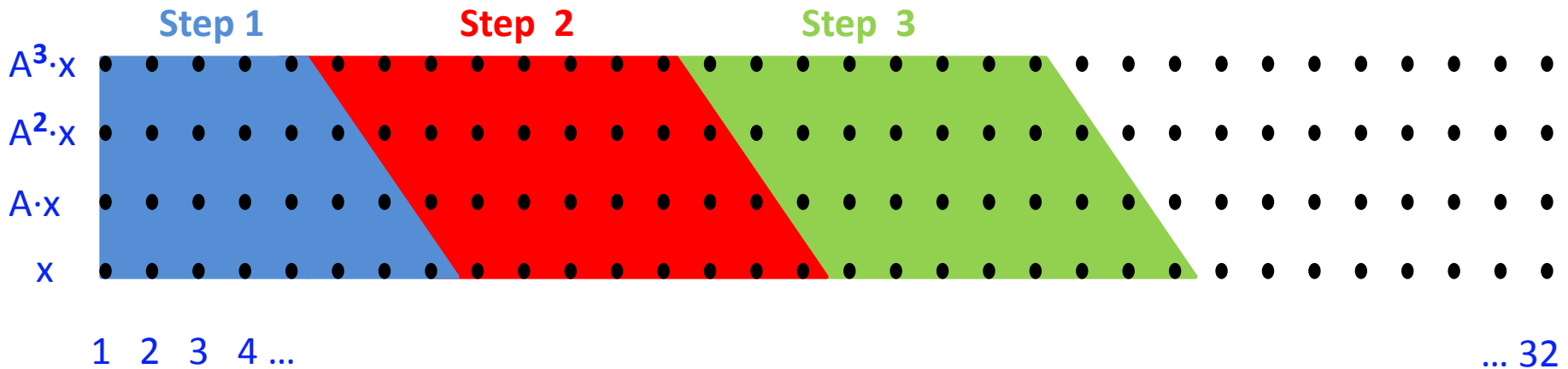


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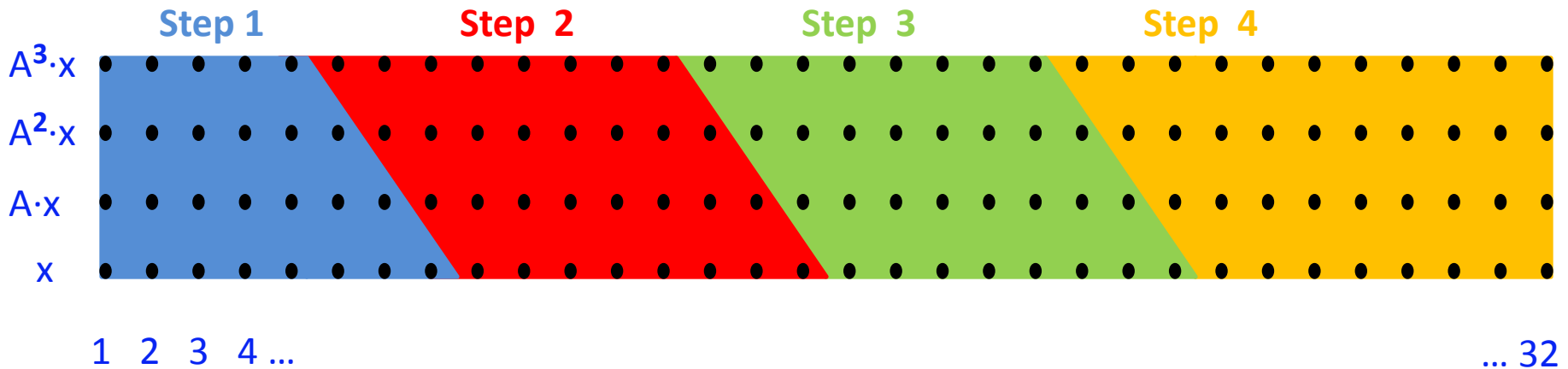


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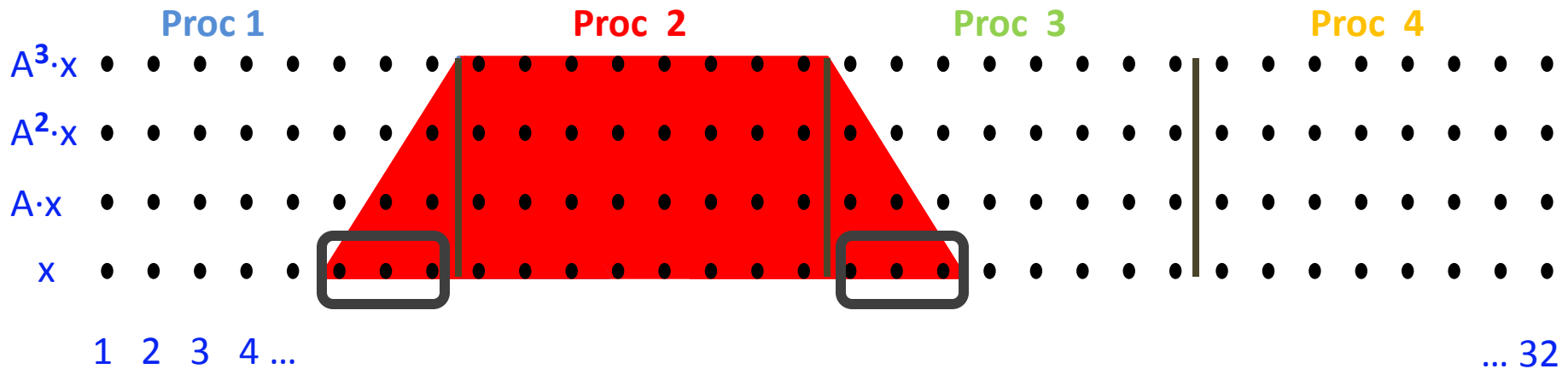


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- Parallel Algorithm

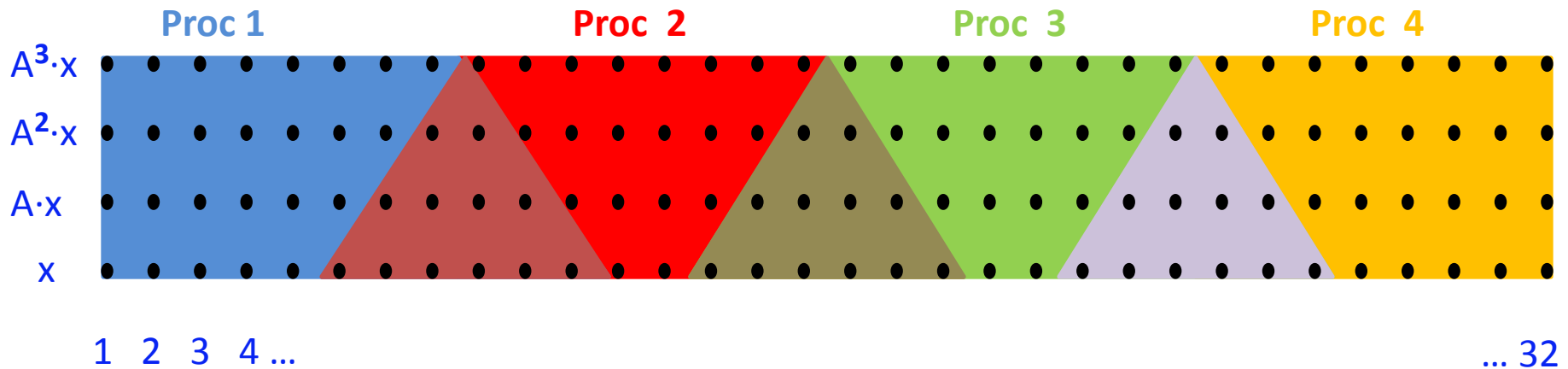


- Example: A tridiagonal, $n=32$, $k=3$
- Each processor communicates once with neighbors

Communication Avoiding Kernels:

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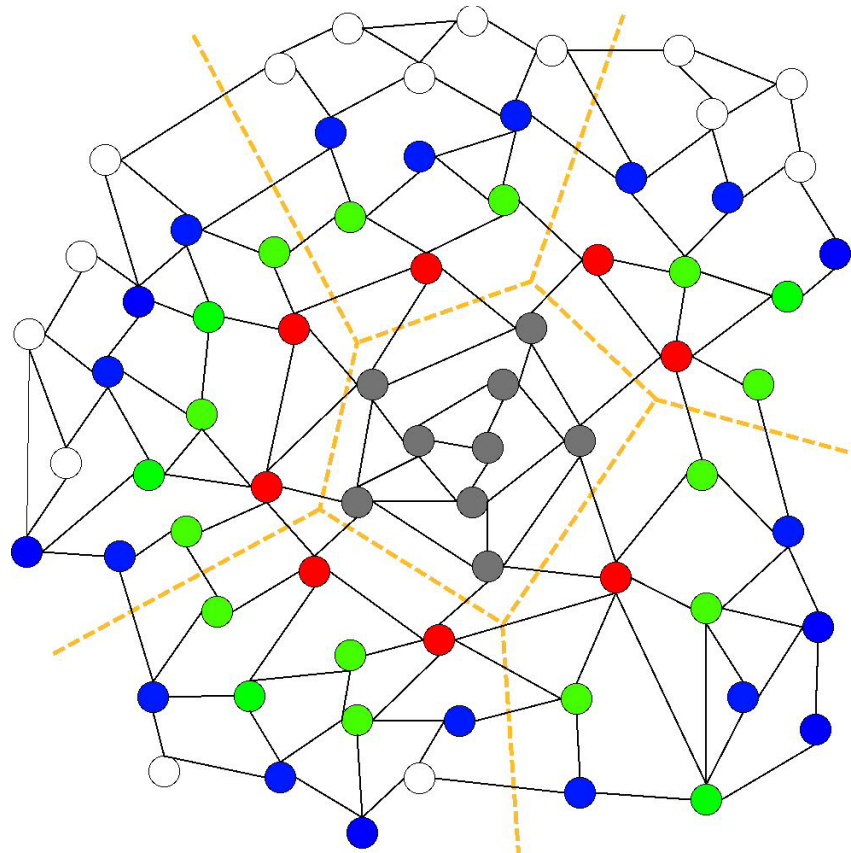
- Replace k iterations of $y = A \cdot x$ with $[Ax, A^2x, \dots, A^kx]$
- Parallel Algorithm



- Example: A tridiagonal, $n=32$, $k=3$
- Each processor works on (overlapping) trapezoid

Communication Avoiding Kernels: The Matrix Powers Kernel : $[Ax, A^2x, \dots, A^kx]$

Same idea works for general sparse matrices



We're hiring!

- Seeking a postdoc to help develop the next versions of LAPACK and ScaLAPACK

7 Dwarfs of High Performance Computing (HPC)

Structured Grid

Dense Matrix

Sparse Matrix

Spectral (FFT)

Particle Methods

Unstructured Grid

Monte Carlo

7 Dwarfs – Are they enough?

Embed
SPEC
DB
Games
ML
CAD
HPC

Structured Grid

Dense Matrix

Sparse Matrix

Spectral (FFT)

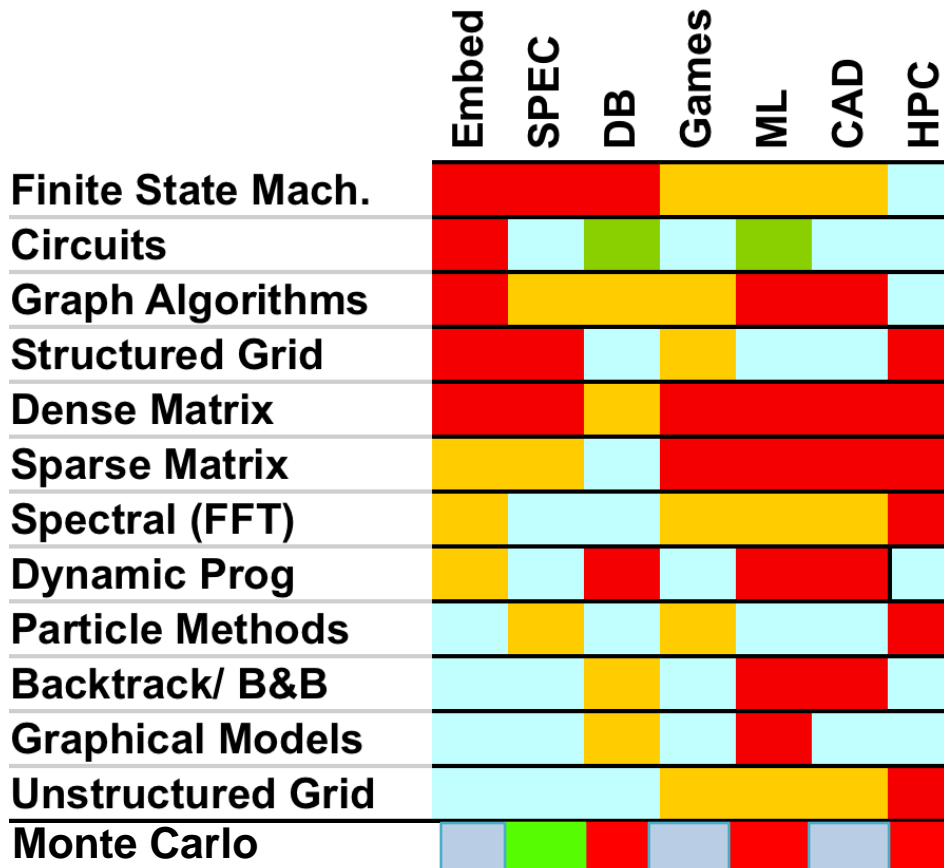
Particle Methods

Unstructured Grid

Monte Carlo






13 Motifs (nee "Dwarf") of Parallel Computing

Popularity: (Red Hot / Blue Cool)



Motifs in ParLab Applications

(Red Hot / Blue Cool)

	Embed	SPEC	DB	Games	ML	CAD	HPC	 Health	 Image	 Speech	 Music	 Browser
1 Finite State Mach.	Red	Red	Red	Yellow	Yellow	Yellow	Light Blue	Light Blue	Light Blue	Light Blue	Light Blue	Light Blue
2 Circuits	Red	Light Blue	Green	Light Blue	Green	Light Blue	Light Blue	Light Blue	Light Blue	Light Blue	Light Blue	Light Blue
3 Graph Algorithms	Red	Yellow	Yellow	Yellow	Red	Red	Light Blue	Red	Light Blue	Red	Green	Green
4 Structured Grid	Red	Red	Light Blue	Yellow	Light Blue	Light Blue	Red	Light Blue	Red	Light Blue	Light Blue	Light Blue
5 Dense Matrix	Red	Red	Yellow	Red	Red	Red	Light Blue	Red	Red	Red	Red	Light Blue
6 Sparse Matrix	Yellow	Yellow	Light Blue	Red	Red	Red	Light Blue	Red	Light Blue	Light Blue	Red	Light Blue
7 Spectral (FFT)	Yellow	Light Blue	Light Blue	Yellow	Yellow	Yellow	Red	Light Blue	Green	Red	Red	Red
8 Dynamic Prog	Yellow	Light Blue	Red	Light Blue	Red	Red	Light Blue	Light Blue	Light Blue	Yellow	Light Blue	Red
9 Particle Methods	Light Blue	Yellow	Light Blue	Yellow	Light Blue	Light Blue	Red	Light Blue	Light Blue	Light Blue	Light Blue	Light Blue
10 Backtrack/ B&B	Light Blue	Light Blue	Yellow	Light Blue	Red	Red	Light Blue	Light Blue	Light Blue	Light Blue	Yellow	Light Blue
11 Graphical Models	Light Blue	Light Blue	Yellow	Light Blue	Red	Light Blue	Light Blue	Light Blue	Light Blue	Light Blue	Red	Light Blue
12 Unstructured Grid	Light Blue	Light Blue	Light Blue	Yellow	Yellow	Yellow	Red	Red	Light Blue	Light Blue	Red	Light Blue

□ What happened to Monte Carlo?

One-sided Factorizations (LU, QR)

- Classical Approach
for $i=1$ to n
 update column i
 update trailing matrix
- #words_moved = $O(n^3)$

- Blocked Approach (LAPACK)
for $i=1$ to n/b
 update **block i of b columns**
 update trailing matrix
- #words moved = $O(n^3/M^{1/3})$

- Recursive Approach
func factor(A)
 if A has 1 column, update it
 else
 factor(left half of A)
 update right half of A
 factor(right half of A)
- #words moved = $O(n^3/M^{1/2})$

- None of these approaches minimizes #messages or works in parallel
- Need another idea

Communication-Avoiding LU:

Use reduction tree, to do “Tournament Pivoting”

$$W^{n \times b} = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix} = \begin{pmatrix} P_1 \cdot L_1 \cdot U_1 \\ P_2 \cdot L_2 \cdot U_2 \\ P_3 \cdot L_3 \cdot U_3 \\ P_4 \cdot L_4 \cdot U_4 \end{pmatrix}$$

Choose b pivot rows of W_1 , call them W_1'
 Ditto for W_2 , yielding W_2'
 Ditto for W_3 , yielding W_3'
 Ditto for W_4 , yielding W_4'

$$\begin{pmatrix} W_1' \\ W_2' \\ W_3' \\ W_4' \end{pmatrix} = \begin{pmatrix} P_{12} \cdot L_{12} \cdot U_{12} \\ P_{34} \cdot L_{34} \cdot U_{34} \end{pmatrix}$$

Choose b pivot rows, call them W_{12}'
 Ditto, yielding W_{34}'

$$\begin{pmatrix} W_{12}' \\ W_{34}' \end{pmatrix} = P_{1234} \cdot L_{1234} \cdot U_{1234}$$

Choose b pivot rows

- Go back to W and use these b pivot rows
 (move them to top, do LU without pivoting)

Collaborators

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Can we do even better?

- Assume $n \times n$ matrices on P processors
- **Why just one copy of data: $M = O(n^2 / P)$ per processor?**

- Recall lower bounds:

$$\begin{aligned}\text{\#words_moved} &= \Omega\left(\frac{(n^3/P)}{M^{1/2}}\right) = \Omega\left(\frac{n^2}{P^{1/2}}\right) \\ \text{\#messages} &= \Omega\left(\frac{(n^3/P)}{M^{3/2}}\right) = \Omega\left(P^{1/2}\right)\end{aligned}$$

Algorithm	Reference	Factor exceeding lower bound for \#words_moved	Factor exceeding lower bound for \#messages
Matrix Multiply	[Cannon, 69]	1	1
Cholesky	ScaLAPACK	$\log P$	$\log P$
LU	[GDX10]	$\log P$	$\log P$
QR	[DGHL08]	$\log P$	$\log^3 P$
Sym Eig, SVD	[BDD11]	$\log P$	$\log^3 P$
Nonsym Eig	[BDD11]	$\log P$	$\log^3 P$

Can we do even better?

- Assume $n \times n$ matrices on P processors
- Why just one copy of data: $M = O(n^2 / P)$ per processor?
- Increase M to reduce lower bounds:
#words_moved = $\Omega((n^3 / P) / M^{1/2}) = \Omega(n^2 / P^{1/2})$
#messages = $\Omega((n^3 / P) / M^{3/2}) = \Omega(P^{1/2})$

Algorithm	Reference	Factor exceeding lower bound for #words_moved	Factor exceeding lower bound for #messages
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Beating $\#words_moved = \Omega(n^2/P^{1/2})$

- $\#words_moved = \Omega((n^3/P)/M^{1/2})$
- If c copies of data, $M = c \cdot n^2/P$, bound decreases by factor $c^{1/2}$
- Can we attain it?
- “3D” Matmul Algorithm on $P^{1/3} \times P^{1/3} \times P^{1/3}$ processor grid
 - $P^{1/3}$ redundant copies of A and B
 - Reduces communication volume to $O((n^2/P^{2/3}) \log(P))$
 - optimal for $P^{1/3}$ copies (more memory can't help)
 - Reduces number of messages to $O(\log(P))$ – also optimal
- “2.5D” Algorithms
 - Extends to $1 \leq c \leq P^{1/3}$ copies on $(P/c)^{1/2} \times (P/c)^{1/2} \times c$ grid
 - Reduces communication volume of Matmul and LU by $c^{1/2}$
 - Reduces comm 83% on 64K proc BG-P, LU&MM speedup 2.6x
- **Distinguished Paper Prize, Euro-Par'11 (E. Solomonik, JD)**

Lower bound for Strassen's fast matrix multiplication

Recall $O(n^3)$ case:

For Strassen's:

For Strassen-like:

Sequential: $\Omega\left(\left(\frac{n}{\sqrt{M}}\right)^{\log_2 8} M\right)$ $\Omega\left(\left(\frac{n}{\sqrt{M}}\right)^{\log_2 7} M\right)$ $\Omega\left(\left(\frac{n}{\sqrt{M}}\right)^{\omega_0} M\right)$

Parallel: $\Omega\left(\left(\frac{n}{\sqrt{M}}\right)^{\log_2 8} \frac{M}{P}\right)$ $\Omega\left(\left(\frac{n}{\sqrt{M}}\right)^{\log_2 7} \frac{M}{P}\right)$ $\Omega\left(\left(\frac{n}{\sqrt{M}}\right)^{\omega_0} \frac{M}{P}\right)$

- Parallel lower bounds apply to 2D (1 copy of data) and 2.5D (c copies)
- Attainable
 - Sequential: usual recursive algorithms, also for LU, QR, eig, SVD,...
 - Parallel: just matmul so far ...
- Talk by Oded Schwartz, Thursday, 5:30pm
- **Best Paper Award, SPAA'11 (Ballard, JD, Holtz, Schwartz)**

Sequential Strong Scaling

	Standard Alg.	CA-CG with SA1	CA-CG with SA2
1D 3-pt stencil	$\gamma \gtrsim \beta$ $\gamma \frac{n}{p} \gtrsim \alpha$	$\gamma s^2 \gtrsim \beta$ $\gamma \frac{s^2 n}{p} \gtrsim \alpha$	$\gamma s \gtrsim \beta$ $\gamma \frac{s^2 n}{p} \gtrsim \alpha$
2D 5-pt stencil	$\gamma \gtrsim \beta$ $\gamma \frac{n^2}{p} \gtrsim \alpha$	$\gamma s \gtrsim \beta$ $\gamma \frac{sn^2}{p} \gtrsim \alpha$	$\gamma \gtrsim \beta$ $\gamma \frac{sn^2}{p} \gtrsim \alpha$
3D 7-pt stencil	$\gamma \gtrsim \beta$ $\gamma \frac{n^3}{p} \gtrsim \alpha$	$\gamma s \gtrsim \beta$ $\gamma \frac{sn^3}{p} \gtrsim \alpha$	$\gamma \gtrsim \beta$ $\gamma \frac{sn^3}{p} \gtrsim \alpha$

Parallel Strong Scaling

	Standard Alg.	CA-CG with PA1
1D 3-pt stencil	$\gamma \frac{n}{p} \gtrsim \beta$ $\gamma \frac{n}{p \log p} \gtrsim \alpha$	$\gamma \frac{n}{p} \gtrsim \beta$ $\gamma \frac{s^2 n}{p \log p} \gtrsim \alpha$
2D 5-pt stencil	$\gamma \frac{n}{\sqrt{p}} \gtrsim \beta$ $\gamma \frac{n^2}{p \log p} \gtrsim \alpha$	$\gamma \frac{n}{\sqrt{p}} \gtrsim \beta$ $\gamma \frac{sn^2}{p \log p} \gtrsim \alpha$
3D 7-pt stencil	$\gamma \frac{n}{p^{1/3}} \gtrsim \beta$ $\gamma \frac{n^3}{p \log p} \gtrsim \alpha$	$\gamma \frac{n}{p^{1/3}} \gtrsim \beta$ $\gamma \frac{sn^3}{p \log p} \gtrsim \alpha$

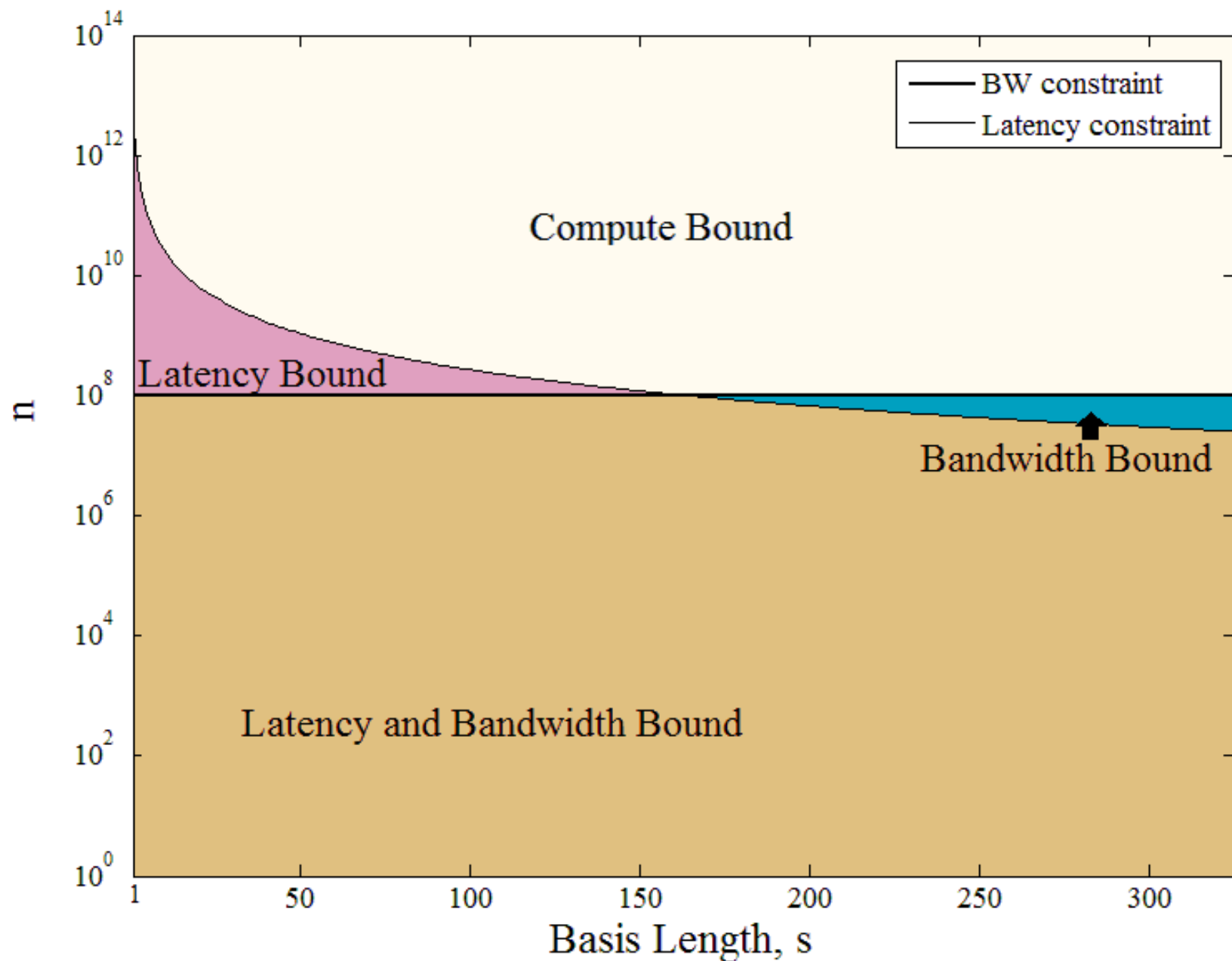
Weak Scaling

- Change p to $x * p$, n to $x^{(1/d)} * n$
 - $d = \{1, 2, 3\}$ for 1D, 2D, and 3D mesh
- Bandwidth
 - Perfect weak scaling for 1D, 2D, and 3D
- Latency
 - Perfect weak scaling for 1D, 2D, and 3D if you ignore the $\log(xp)$ factor in the denominator
 - Makes constraint on alpha harder to satisfy

Performance Model Assumptions

- Plot for Parallel Algorithm for 1D 3-pt stencil
- Exascale machine parameters:
 - 100 GB/sec interconnect BW
 - 1 microsecond network latency
 - 2^{28} cores
 - .1 ns per flop (per core)

Predicted CA-CG Performance on Exascale Machines



Observations

- $s = 1$ are the constraints for the standard algorithm
 - Standard algorithm is communication bound if $n < \sim 10^{12}$
- For $10^8 < \sim n < \sim 10^{12}$, we can theoretically increase s such that the algorithm is no longer communication bound
 - In practice, high s values have some complications due to stability, but even $s \sim 10$ can remove communication bottleneck for matrix sizes $\sim 10^{10}$